Math 213, Spring 2015, Exam 1

1. Consider the vectors $\boldsymbol{u} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, and $\boldsymbol{w} = -\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$.
   a. Find the area of the parallelogram determined by $\boldsymbol{u}$ and $\boldsymbol{v}$.
   $$\vec{u} \times \vec{v} = 2\vec{\imath} - 3\vec{\jmath} - \vec{k}$$
   $$|\vec{u} \times \vec{v}| = \sqrt{14}$$
   b. Find the volume of the parallelepiped determined by $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.
   $$|(\vec{u} \times \vec{v}) \cdot \vec{w}| = |1| = 1$$
   c. Find the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.
   $$\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}\right) = \arccos\left(-\frac{1}{\sqrt{6}\sqrt{14}}\right) = \arccos\left(\frac{1}{\sqrt{84}}\right)$$

2. Find an equation for the plane that passes through the point $(1,2,3)$ parallel to $\boldsymbol{u} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$.
   $$\vec{n} = \vec{u} \times \vec{v} = 7\vec{\imath} - 3\vec{\jmath} - 5\vec{k}$$
   $\overrightarrow{P_0P} \cdot \vec{n} = 0, \quad < x-1, y-x, z-3 > \cdot < 7,-3,-5 > = 0, \quad 7x - 3y - 5z + 14 = 0$

3. The line L: $x = 3 + 2t$, $y = 2t$, $z = t$ intersects the plane $x + 3y - z = -4$ in a point P. Find the coordinates of P and find equations of the line in the plane through P and perpendicular to L.
   $(3+2t) + 3(2t) - t = -4, \qquad t = -1, \qquad P = (1,-2,1)$
   $$\vec{n} = < 1,3,-1 >$$
   $$\vec{v} = \vec{n} \times < 2,2,1 > = < 5,-3,-4 >$$
   $x = 5t + 1, y = -3t - 2, z = -6t - 1$

4. Find the distance between the line L_1 through the points $A(1,0,-1)$ and $B(-1,1,0)$ and the line L_2 through the points $C(3,1,-1)$ and $D(4,5,-2)$.
   $\overrightarrow{v_1} = < -2,1,1 >, \overrightarrow{v_2} = < 1,4,-1 >, \qquad \vec{n} = \overrightarrow{v_1} \times \overrightarrow{v_2} = < -5,-1,-9 >$
   $$dist = \left|proj_{\vec{n}}\overrightarrow{AC}\right| = \frac{\left|\overrightarrow{AC} \cdot \vec{n}\right|}{|\vec{n}|} = \frac{11}{\sqrt{107}}$$

5. Sketch the graph of $\boldsymbol{r}(t) = \sin^2 t\, \boldsymbol{i} + \cos^2 t\, \boldsymbol{j} - \boldsymbol{k}$, $0 \le t \le \frac{\pi}{2}$, and show that it lies in the plane $3x + 3y - 5z = 8$.
   Line segment from $(0,1,-1)$ to $(1,0,-1)$,
   $$3x + 3y - 5z = 3\sin^2 t + 3\cos^2 t - (-5) = 8$$

6. Consider the vector-valued function $\boldsymbol{r}(t) = \sin t\, \boldsymbol{i} + 5\cos t\, \boldsymbol{j}$, $t \ge 0$.
   a. Sketch the image of $\boldsymbol{r}(t)$.
   Ellipse $x^2 + \frac{y^2}{25} = 1$
   b. Sketch $\boldsymbol{r}\left(\frac{\pi}{4}\right)$ and $\boldsymbol{r}'\left(\frac{\pi}{4}\right)$ on the image of $\boldsymbol{r}(t)$.
   $\vec{r}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\vec{\imath} + \frac{5\sqrt{2}}{2}\vec{\jmath}, \qquad \vec{r}'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\vec{\imath} - \frac{5\sqrt{2}}{2}\vec{\jmath}$
   c. Find the equations of the line tangent to the curve $\boldsymbol{r}(t)$ at $t = \frac{\pi}{4}$.

$$\vec{r}(t) = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}t\right)\vec{\imath} + \left(\frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}t\right)\vec{\jmath}$$

d. Suppose that a particle follows the path $\boldsymbol{r}(t)$ until it flies off on a tangent at $t = \frac{\pi}{4}$. Find the coordinates of the particle at $t = \frac{3\pi}{4}$.

$$\vec{r}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = \vec{r}\left(\frac{\pi}{2}\right) = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(\frac{\pi}{2}\right)\right)\vec{\imath} + \left(\frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}\left(\frac{\pi}{2}\right)\right)\vec{\jmath}$$

7.

a. Identify and sketch the surface $x^2 + y^2 + z^2 - 8x + 6z = 0$.
$(x-4)^2 + y^2 + (z+3)^2 = 25$, sphere of radius 5 centered at $(4,0,-3)$

b. Sketch the circular paraboloid $x = y^2 + z^2$. Give the parametric equations for the curve of intersection of this paraboloid with the plane $x = 1$.